EE 103 Lecture 11

April 26, 2017

Fourier series of $x(t)$, where

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_0 = X_0 \frac{T}{T_0}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} X_0 e^{-jk\omega_0 t}\, dt$$

$$= \frac{X_0}{T_0} \frac{e^{-jk\omega_0 t}}{-jk\omega_0}\Bigg|_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{X_0}{T_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{X_0}{T_0} \frac{1}{-jk\omega_0}\left[e^{-jk\omega_0\frac{T}{2}} - e^{+jk\omega_0\frac{T}{2}}\right]$$

$$= \frac{X_0}{T_0} \frac{1}{k\omega_0} 2 \frac{e^{+jk\omega_0\frac{T}{2}} - e^{-jk\omega_0\frac{T}{2}}}{2j}$$

$$= \frac{X_0}{T_0} \frac{1}{k\omega_0} 2 \sin k\omega_0\frac{T}{2}$$

$$= \frac{X_0}{T_0} T \left(\frac{\sin k\omega_0 \frac{T}{2}}{k\omega_0 \frac{T}{2}}\right) = \frac{X_0 T}{T_0} \operatorname{sinc}\left(k\omega_0\frac{T}{2}\right)$$

$$\frac{\sin x}{x}$$

Trigonometric expression of periodic $x(t)$

$$x(t) = c_0 + \sum_{k=1}^{\infty}\left(A_k \cos k\omega_0 t + B_k \sin k\omega_0 t\right) \qquad (1)$$

where
$$A_k = \frac{2}{T_0}\int_{T_0} x(t) \cos k\omega_0 t\, dt \qquad (2)$$

$$B_k = \frac{2}{T_0}\int_{T_0} x(t) \sin k\omega_0 t\, dt \qquad (3)$$

Proof of (2): From (1) & (2)

$$A_k = \frac{2}{T_0}\int_{T_0} \boxed{c_0 + \sum_{k=1}^{\infty}\left(A_k \cos k\omega_0 t + B_k \sin k\omega_0 t\right)} \cdot \cos k\omega_0 t\, dt$$

$$\int_{T_0} c_0 \cos k\omega_0 t\, dt = 0$$

In fact, nonzero only for $\int_{T_0} A_k \cos^2 k\omega_0 t\, dt$

$$\frac{2}{T_0}\int A_k \cos^2 k\omega_0 t\, dt \qquad \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$= \frac{2A_k}{T_0}\int_{T_0}\left(\frac{\cos 2k\omega_0 t + 1}{2}\right) dt$$

$$= \frac{\cancel{2}A_k}{\cancel{T_0}} \frac{1}{\cancel{2}}\left[\underbrace{\int_{T_0} \cos 2k\omega_0 t}_{=0} + \cancel{T_0}\right]$$

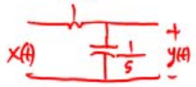
$$= \boxed{A_k}$$

You can show for B_k using

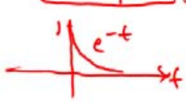
$$\sin^2 k\omega_0 t = 1 - \cos^2 k\omega_0 t$$
$$= 1 - \left(\frac{\cos 2k\omega_0 t + 1}{2}\right) = \frac{1}{2}\left(1 - \cos 2k\omega_0 t\right)$$

$x(t) = e^{st}$ $\qquad$ $y(t) = H(s)\, e^{st}$

$H(s)$



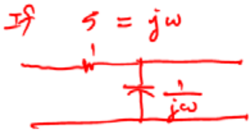
$$\frac{\frac{1}{s}}{\frac{1}{s} + 1} = \frac{1}{s+1}$$



$\rightarrow e^{-t} = h(t)$

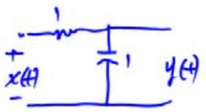
$\delta(t) \longrightarrow h(t)$

If $s = j\omega$



$$\frac{1}{1 + j\omega} = H(j\omega)$$

$x(t) = e^{j\omega t}$ $\qquad$ $y(t) = \underline{H(j\omega)}\, e^{j\omega t}$

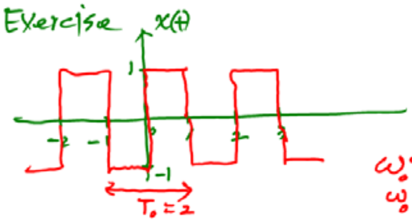


$$x(t) = 1\,i + y$$
$$i = 1\,\frac{dy}{dt}$$
$$x(t) = \frac{dy}{dt} + y$$
$$X(t) = e^{st} \qquad y(t) = Ye^{st}$$
$$e^{st} = Y\,s e^{st} + Y e^{st}$$
$$= Y(s+1)\,e^{st}$$
$$Ye^{st} = y(t) = \frac{e^{st}}{s+1} = \boxed{\frac{1}{s+1}}\, e^{st} \cdot x(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$
$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$
$$y(t)_{\text{steady state}} = \sum_{k=-\infty}^{\infty} \underbrace{C_{kx} H(jk\omega_0)}_{= C_{ky}} e^{jk\omega_0 t}$$
$$\boxed{C_{ky} = C_{kx} H(jk\omega_0)}$$
$$|C_{ky}| \angle C_{ky} = |C_{kx}| \angle C_{kx}\, |H(jk\omega_0)| \angle H(jk\omega_0)$$
$$|C_{ky}|\, e^{j\theta_{ky}} = |C_{kx}|\, e^{j\theta_{kx}}\, |H(jk\omega_0)|\, e^{j\phi_k}$$

$$y_{ss} = C_{0y} + \sum_{k=1}^{\infty} 2|C_{ky}| \cos\big(k\omega_0 t + \underbrace{\theta_{kx} + \phi_k}_{\theta_{ky}}\big)$$

Exercise



$$\omega_0 T_0 = 2\pi$$
$$\omega_0 = \pi$$
$$A_k = \frac{2}{T_0 = 2}\left[\int_0^1 1 \cos k\pi t\, dt + \int_1^2 (-1) \cos k\pi t\, dt\right]$$
$$= \frac{\sin k\pi t}{k\pi}\Big|_0^1 - \frac{\sin k\pi t}{k\pi}\Big|_1^2 = 0 - 0 = 0 \ \forall k$$

$$B_k = \left[\frac{2}{T_0 = 2}\right]\left[\int_0^1 \sin k\pi t\, dt + \int_1^2 (-1) \sin k\pi t\, dt\right]$$
$$= \frac{-\cos k\pi t}{k\pi}\Big|_0^1 + \frac{\cos k\pi t}{k\pi}\Big|_1^2$$
$$= \frac{1}{k\pi}\left[1 - \cos k\pi + \cos 2k\pi - \cos k\pi\right]$$
$$k = \text{odd} \qquad \frac{1}{k\pi}\left[2 + 1 + 1\right] = \frac{4}{k\pi}$$
$$k = \text{even} \qquad \frac{1}{k\pi}\left[1 - 1 + 1 - 1\right] = 0$$
$$x(t) = \frac{4}{\pi}\left[\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \cdots\right]$$

(Frequency) spectrum

$$x(t) = \sum_{k=-\infty} C_{kx} e^{jk\omega_0 t}$$
$$C_{kx} = |C_{kx}|\, e^{j\theta_{kx}}$$

$$x(t) \longleftrightarrow C_{kx} \ \text{in Fourier Series}$$
$$x(t) = \sum_{k=-\infty}^{\infty} C_{kx}\, e^{+jk\omega_0 t}$$
$$C_{kx} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

For $x(t - t_0)$ delay by t_0

$$x(t - t_0) = \hat{x}(t)$$
$$C_{k\hat{x}} = \frac{1}{T_0}\int_{T_0} x(t - t_0)\, e^{-jk\omega_0 t}\, dt$$
$$= \frac{1}{T_0}\int_{T_0} x(\underbrace{t - t_0}_{t'})\, e^{-jk\omega_0 (t - t_0)}\, e^{-jk\omega_0 t_0}\, d(t - t_0)$$
$$= e^{-jk\omega_0 t_0}\, \frac{1}{T_0}\int_{T_0} x(t')\, e^{-jk\omega_0 t'}\, dt'$$

Quiz 3

$x(t) = 5 + \underline{\sin 2t} + 1\cos 4t$

Find c_k $\quad 0, 1, 2$

$\omega_0 = 2 \qquad T_0 = \pi \quad (\omega_0 T_0 = 2\pi)$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$= \sum_{k=-2}^{2} c_k e^{j2kt} + \cdots$$

$$= c_{-2} e^{-j4t} + \underbrace{c_{-1} e^{-j2t} + c_0 + c_1 e^{+j2t}}_{2c_1 \cos 2t = 2c_1 \sin(2t+90^\circ)} + c_2 e^{+j4t} + \cdots$$

$c_0 = 5$

$\boxed{2c_2}\cos 4t$

$c_2 = \frac{1}{2}$

$2c_1 \underline{\cos 2t} = \sin 2t$

$= \sin(2t + 90^\circ)$

$2c_1 \sin 2t\, e^{j90^\circ} = 1 \sin 2t \rightarrow c_1 = \frac{1}{2j}$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(2t + 90^\circ) = \sin 2t \underbrace{\cos 90^\circ}_{=0} + \cos 2t \underbrace{\sin 90^\circ}_{=1}$

$= \cos 2t$